Draft version June 10, 1999
Preprint typeset using LaTeX style emulateapj

ECCENTRIC BEHAVIOR OF DISK GALAXIES

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Draft version June 10, 1999

ABSTRACT

A theory is developed for the dynamics of eccentric perturbations [$\propto \exp(\pm i\phi)$] of a disk galaxy residing in a spherical dark matter halo and including a spherical bulge component. The disk is represented as a large number N of rings with shifted centers *and* with perturbed azimuthal matter distributions. Account is taken of the dynamics of the shift of the matter at the galaxy's center which may include a massive black hole. The gravitational interactions between the rings and between the rings and the center is fully accounted for, but the halo and bulge components are treated as passive gravitational field sources. Equations of motion and a Lagrangian are derived for the ring+center system, and these lead to total energy and total angular momentum constants of the motion.

We first study the eccentric motion of a disk consisting of two rings of different radii but equal mass, $M_d/2$. For small M_d the two rings are stable, but for M_d larger than a threshold value the rings are unstable with a dynamical time-scale growth. For M_d sufficiently above this threshold, the instability acts to decrease the angular momentum of the inner ring, while increasing that of the outer ring. The instability results from the merging positive and negative energy modes with increasing M_d. Secondly, we analyze the eccentric motion of one ring interacting with a radially shifted central mass. In this case instability sets in above a threshold value of the central mass (for a fixed ring mass), and it acts to increase the angular momentum of the central mass (which therefore rotates in the direction of the disk matter), while decreasing the angular momentum of the ring.

Thirdly, we study the eccentric dynamics of a disk with an exponential surface density distribution represented by a large number of rings. The inner part of the disk is found to be strongly unstable. Angular momentum of the rings is transferred outward *and* to the central mass if present, and a trailing one-armed spiral wave is formed in the disk. Fourthly, we analyze a disk with a modified exponential density distribution where the density of the inner part of the disk is reduced. In this case we find much slower, linear growth of the eccentric motion. A trailing one-armed spiral wave forms in the disk and becomes more tightly wrapped as time increases. The motion of the central mass if present is small compared with that of the disk.

1. INTRODUCTION

Although studies of spiral galaxies commonly assume an axisymmetric equilibrium state, possibly perturbed by spiral arms, there is growing evidence that many galaxies lack this symmetry. Based on optical appearance, approximately 30% of disk galaxies exhibit significant "lopsidedness" (Rix & Zaritsky 1995; Kornreich, Haynes, & Lovelace 1998) which supports the early findings of Baldwin, Lynden-Bell, & Sancisi (1980). As many as $\sim 50\%$ of spiral galaxies show departures from the expected symmetric two–horned global HI line profile (Richter & Sancisi 1994; Haynes *et al.* 1998). Furthermore, HI maps of several galaxies, for example, NGC 3631 (Knapen 1997), NGC 5474 (Rownd, Dickey, & Helou 1994), and NGC 7217 (Buta *et al.* 1995), have revealed offsets between the optical centers of light and their kinematic centers. Two examples of kinematically lopsided galaxies have recently be discussed by Swaters *et al.* (1998). Other recent observations such as the *Hubble Space Telescope* observations of the nucleus of M31 (Lauer *et al.* 1993) and the Kitt Peak 0.9 meter observations of NGC 1073 (Kornreich *et al.* 1998) indicate that even the optical center of light may be displaced from the center of the optical isophotes of the major part of the galaxy in a significant fraction of cases.

The origin of the lopsideness in disk galaxies remains uncertain. Baldwin *et al.* (1980) proposed a simple kinematic model in which different rings of the galaxy, assumed non-interacting, are initially shifted from their centered equilibrium positions. The shifted rings precess in the overall gravitational potential in a direction opposite to that of the mass motion. Because the precession rate decreases in general with radial distance, an initial disturbance tends to "wind up" into a leading spiral arm in a time appreciably less than the Hubble time. Miller and Smith (1988, 1992) have made extensive computer simulation studies of the unstable eccentric motion of matter in the nuclei of galaxies which they suggest is pertinent to the off-center nuclei observed in a number of galaxies, for example, M31, M33, and M101.

Understanding of the origin of the observed disk asymmetries is important because it provides clues to ongoing accretion of gas and the distribution of dark, unseen mass in the halo. Schoenmakers, Franx, and de Zeeuw (1997), interpret optical asymmetry as an indicator of asymmetry in the overall galactic potential, and therefore an indicator of the spatial distribution of the dark matter in a galaxy which may have a triaxial distribution. By analyzing the spiral components present in the surface brightness or H I distribution, the velocity gradient, and therefore, the shape of the gravitational potential, may be uncovered. Jog (1997) has studied of the orbits of stars and gas in a lopsided potential, and shows that lopsided potentials arising from disks alone are not self-consistent; rather, a stationary lopsided disk may be responding to asymmetries in the halo.

Zaritsky and Rix (1997) proposed that optical lopsidedness arises from tidal interactions and/or minor mergers. Such mergers are often suggested as the most likely contributors to galaxy asymmetry, even when no interacting companions are evident. While galaxies such as NGC 5474 are well-known to be under the tidal influence of neighbors, the apparently long-lived kinematic offsets of other relatively isolated objects, and the common asymmetries in flocculent (as opposed to tidally induced "grand design") spiral galaxies, are not explained by simple tidal interaction models, which produce only transient asymmetric features.

A further possibility is that an optical disk may be in a quasi-stationary lopsided state in a symmetric potential, as discussed by Syer and Tremaine (1996). In this model, gaseous and stellar matter swirl about the minimum of the halo potential in a state not fully relaxed. The result is a lopsided flow within a symmetric mass distribution. Numerical simulations of this situation have been done by Levine and Sparke (1998) using a gravitational $N-$body tree-code method (see Barnes & Hut 1989) for disk galaxies shifted from the center of the main halo potential. The results are suggestive of lopsidedness with large lifetimes. An $N-$body simulation study of a rotating spheroidal stellar system including the dynamics of a massive central object by Taga and Iye (1998a) indicates that the central object goes into a long lasting oscillation similar to those found earlier by Miller and Smith (1988, 1992) and which may explain asymmetric structures observed in M31 and NGC 4486B. A linear stability analysis of a self-gravitating fluid disk including a massive central object also by Taga and Iye (1998b) indicates a linear instability (Taga & Iye 1998b). We comment on the relation of this work to the present study in the conclusions section of this work.

Here, we develop a theory of the dynamics of eccentric perturbations of a disk galaxy residing in a spherical dark matter halo. We represent the disk as a large number N of rings as suggested by Baldwin *et al.* (1980) (and Lovelace (1998) for the treatment of disk warping). In contrast with Baldwin *et al.*, the gravitational interactions between the rings is fully accounted for. We show that for general eccentric perturbations, the centers of the rings are shifted *and* the azimuthal distribution of matter in the rings is perturbed. The ring representation is analogous to the approach of Contopoulos and Grøsbal (1986, 1988), where self-consistent galaxy models are constructed from a finite set of stellar orbits.

Section 2 develops a theory for treating eccentric perturbations of a disk galaxy. The assumed equilibrium is first discussed (§2.1), and a description of the disk perturbations is developed (§2.2). The representation of the disk in terms of a finite number N of rings is presented (§2.3), and the ring equations of motion are derived (§2.4). We renormalize the ring equations so as to reduce the nearest ring interactions (§2.5). The dynamics and influence of the displacement of the center of the galaxy, which may include a massive black hole, is discussed separately (§2.6). We obtain an energy constant of the motion for the dynamical equations (§2.7), the Lagrangian, and the conserved total canonical angular momentum (§2.8).

We discuss the nature of the precession of a single ring in §3. In §4 we study the eccentric motion of a disk consisting of two rings and show that this motion is unstable for sufficiently large ring masses. In §5 we study the eccentric motion of one ring including the radial shift of the central mass and show that this situation is unstable for sufficiently large mass of the center and/or of the ring. Section 6 presents numerical results for the eccentric dynamics of disk of many rings including the radial shift of the central mass. Section 7 summarizes the conclusions of this work.

2. THEORY

2.1. *Equilibrium*

The equilibrium galaxy is assumed to be axisymmetric and to consist of a thin disk of stars and gas and a spheroidal distributions consisting of a bulge component and a halo of dark matter. We use an inertial cylindrical (r, ϕ, z) and Cartesian (x, y, z) coordinate systems with the disk and halo equatorial planes in the $z = 0$ plane. The total gravitational potential is written as

$$\Phi(r, z) = \Phi_d + \Phi_b + \Phi_h \ , \tag{1}$$

where Φ_d is the potential due to the disk, Φ_b is due to the bulge, and Φ_h is that for the halo. The galaxy may have a central massive black hole of mass M_{bh} in which case a term $\Phi_{bh} = -GM_{bh}/\sqrt{r^2 + z^2}$ is added to the right-hand side of (1). The particle orbits in the equilibrium disk are approximately circular with angular rotation rate $\Omega(r)$, where

$$\Omega^2(r) = \frac{1}{r}\frac{\partial \Phi}{\partial r}\bigg|_{z=0} = \Omega_d^2 + \Omega_b^2 + \Omega_h^2 \ . \tag{2}$$

The equilibrium disk velocity is $\mathbf{v} = r\Omega(r)\hat{\boldsymbol{\phi}}$. A central black hole is accounted for by adding the term $\Omega_{bh}^2 = GM_{bh}/r^3$ to the right-hand side of (2).

The surface mass density of the (optical) disk is taken to be $\Sigma_d = \Sigma_{d0}\exp(-r/r_d)$ with Σ_{d0} and r_d constants and $M_d = 2\pi r_d^2 \Sigma_{d0}$ the total disk mass. The potential due to this disk matter is

$$\Phi_d(r, 0) = -\ \frac{GM_d}{r_d} R[I_0(R)K_1(R) - I_1(R)K_0(R)] \ ,$$

and the corresponding angular velocity is

$$\Omega_d^2 = \frac{1}{2}\frac{GM_d}{r_d^3}\left[I_0(R)K_0(R) - I_1(R)K_1(R)\right] \ , \tag{3}$$

where $R \equiv r/(2r_d)$ and the $I's$ and $K's$ are the usual modified Bessel functions (Freeman 1970; Binney & Tremaine

Fig. 1.— Sample disk rotation curve $v_\phi(r)$ for the values $M_d = 6 \times 10^{10} M_\odot$ and $r_d = 4$ kpc for the disk, $M_b = 5 \times 10^9 M_\odot$ and $r_b = 1$ kpc for the bulge, and $v_h = 250$ km/s and $r_h = 5$ kpc for the halo, using expressions given in §2.1.

1987, p.77). Typical values are $M_d = 6 \times 10^{10}\mathrm{M}_\odot$ and $r_d = 4$ kpc. For these values, $v_d \equiv \sqrt{GM_d/r_d} \approx 255$ km/s.

The potential due to the bulge component is taken as a Plummer model

$$\Phi_b = -\frac{GM_b}{(r_b^2 + r^2 + z^2)^{1/2}} ,$$

where M_b is the mass of the bulge and r_b is its characteristic radius (Binney & Tremaine 1987, p.42). We have

$$\Omega_b^2 = \frac{GM_b}{(r_b^2 + r^2)^{3/2}} . \tag{4}$$

Typical values are $M_b = 10^{10}\mathrm{M}_\odot$ and $r_b = 1$ kpc, and for these values $v_b \equiv \sqrt{GM_b/r_b} \approx 208$ km/s.

The potential of the halo is taken to be

$$\Phi_h = \frac{1}{2} v_h^2 \ \ln(r_h^2 + r^2 + z^2) ,$$

where $v_h =$ const is the circular velocity at large distances and $r_h =$ const is the core radius of the halo. We have

$$\Omega_h^2 = \frac{v_h^2}{r_h^2 + r^2} . \tag{5}$$

Typical values are $v_h \sim 200 - 300$ km/s and $r_h \sim 2 - 20$ kpc. Figure 1 shows an illustrative rotation curve.

2.2. *Perturbations*

We treat the disk as fluid and use a Lagrangian representation for the perturbation as developed by Frieman and Rotenberg (1960). The position vector $\mathbf{r}$ of a fluid element which at $t = 0$ was at $\mathbf{r}_0$ is given by

$$\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\xi}(\mathbf{r}_0, t) . \tag{6}$$

That is, $\mathbf{r}_0(t)$ is the unperturbed and $\mathbf{r}(t)$ the perturbed orbit of a fluid element. This description is applicable to *both* the disk gas and the disk stars which are in approximately laminar motion with circular orbits. The perturbations of the halo and bulge are assumed negligible compared with that of the disk. For these approximately spheroidal components, the particle motion is highly nonlaminar with criss-crossing orbits with the result that their response "averages out" the disk perturbation.

Further, the perturbations are assumed to consist of small in-plane displacements or shifts of the disk matter,

$$\boldsymbol{\xi} = \xi_r \hat{\mathbf{r}} + \xi_\phi \hat{\boldsymbol{\phi}} , \tag{7}$$

with azimuthal mode number $m = 1$. That is, ξ_r and ξ_ϕ have ϕ-dependences proportional to $\exp(i\phi)$.

From equation (6), we have $\mathbf{v}(\mathbf{r}, t) = \mathbf{v}_0(\mathbf{r}_0, t) + \partial\boldsymbol{\xi}/\partial t + (\mathbf{v} \cdot \nabla)\boldsymbol{\xi}$. The Eulerian velocity perturbation is $\delta\mathbf{v}(\mathbf{r}, t) \equiv \mathbf{v}(\mathbf{r}, t) - \mathbf{v}_0(\mathbf{r}, t)$. Therefore,

$$\delta\mathbf{v}(\mathbf{r}, t) = \frac{\partial\boldsymbol{\xi}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\,\mathbf{v} . \tag{8}$$

The components of this equation are

$$\begin{aligned} \delta v_r &= \mathcal{D}\,\xi_r , \\ \delta v_\phi &= \mathcal{D}\,\xi_\phi - r\Omega'\,\xi_r , \end{aligned} \tag{9}$$

where

$$\mathcal{D} \equiv \frac{\partial}{\partial t} + \Omega(r)\frac{\partial}{\partial\phi} ,$$

and $\Omega' \equiv \partial\Omega/\partial r$.

The main equation of motion is

$$\frac{d\ \delta\mathbf{v}}{dt} = \delta\mathbf{F} = -\nabla\delta\Phi , \tag{10}$$

where $\delta\mathbf{F}$ is the perturbation in the gravitational force (per unit mass), and $\delta\Phi$ is the perturbation of the gravitational potential. The pressure force contribution is small compared to $\delta\mathbf{F}$ by a factor $(v_{th}/v_\phi)^2 \ll 1$ and is neglected, where v_{th} is the 'thermal' spread of the velocities of the disk matter. Also, note that

$$\frac{d\ \delta\mathbf{v}}{dt} = \frac{\partial\ \delta\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\delta\mathbf{v} + (\delta\mathbf{v} \cdot \nabla)\,\mathbf{v} . \tag{11}$$

The components of (11) give

$$\begin{aligned}\left(\frac{d\ \delta\mathbf{v}}{dt}\right)_r &= \mathcal{D}\delta v_r - 2\Omega\delta v_\phi\ ,\\ &= (\mathcal{D}^2 + 2\Omega r\Omega')\ \xi_r - 2\Omega\mathcal{D}\ \xi_\phi\ ,\\ \left(\frac{d\ \delta\mathbf{v}}{dt}\right)_\phi &= \mathcal{D}\delta v_\phi + (\kappa_r^2/2\Omega)\delta v_r\ ,\\ &= \mathcal{D}^2\ \xi_\phi + 2\Omega\mathcal{D}\ \xi_r\ , \end{aligned} \qquad (12)$$

where $\kappa_r^2 \equiv (1/r^3)d(r^4\Omega^2)/dr$ is the radial epicyclic frequency (squared).

The perturbation of the surface mass density of the disk obeys

$$\frac{\partial\ \delta\Sigma}{\partial t} = -\nabla\cdot(\Sigma\ \delta\mathbf{v} + \delta\Sigma\ \mathbf{v})\ ,$$

where $\Sigma(r)$ is the surface density of the equilibrium disk. Because $\nabla\cdot(\Sigma\ \mathbf{v}_0) = 0$, this equation implies

$$\delta\Sigma = -\nabla\cdot(\Sigma\ \boldsymbol{\xi})\ . \qquad (13)$$

The perturbation of the gravitational potential is given by

$$\delta\Phi(\mathbf{r},t) = -G\int d^2r'\ \frac{\delta\Sigma(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\ , \qquad (14)$$

where the integration is over the surface area of the disk.

2.3. *Ring Representation*

We represent the disk by a finite number N of radially shifted plane circular rings. The matter distribution around each ring is also perturbed. An elliptical distortion of a ring corresponds to an $m = \pm 2$ which is not considered here. This description is *general* for small shifts $(dr/r)^2 \ll 1$, where the linearized equations are applicable (Lovelace 1998). Of course, the orbit of a single perturbed particle is not in general closed in the inertial frame used. However, the orbit *is* closed in an appropriately rotating frame, and this rotation rate is simply the angular precession frequency of the ring ω discussed below (Baldwin *et al.* 1980). The disk is assumed geometrically thin.

For the equilibrium disk we take

$$\Sigma(r) = \sum_{j=1}^{N}\frac{M_j}{2\pi r\sqrt{2\pi}\Delta r_j}\ \exp\left[-\frac{(r-r_j)^2}{2\Delta r_j^2}\right]\ , \qquad (15)$$

where M_j is the mass of the j^{th} ring, r_j is its radius with $0 < r_1 < r_2\ ...\ r_N$, and $\Delta r_j \ll r_j$ is its width. The motion of the central part of the disk ($r < r_1$) is treated separately in §2.6

A physical choice for the rings distribution will have the ring spacing of the order of the disk thickness, $r_{j+1} - r_j = \mathcal{O}(\Delta z)$. Further, we assume $(r_{j+1} - r_j)^2 \ll r_{j+1}r_j$ in order to simplify the calculation of the ring interaction as discussed in Appendix A. For example, a possible choice is $r_j = r_1 + (j-1)\delta r$ with $r_1 = 1$ kpc, $\delta r = 0.5$ kpc, and $M_j = 2\pi r_j\delta r\Sigma_d(r_j)$. For $\Delta r_j = \delta r/\sqrt{2\ln 2} \approx \delta r/1.177$, the profile of a ring falls to half its maximum value at δr so that equation (15) gives a fairly smooth representation of $\Sigma_d(r)$.

The perturbation in the disk's surface density is

$$\delta\Sigma(r,\phi,t) = \sum\delta\Sigma_j\ ,\quad \delta\Sigma_j = -\nabla\cdot(\Sigma_j\boldsymbol{\xi})\ , \qquad (16)$$

where $\Sigma_j \equiv (M_j/2\pi r\sqrt{2\pi}\Delta r_j)\exp[-(r-r_j)^2/(2\Delta r_j^2)]$.

We can express different moments of the perturbed disk in terms of the rings. For example, the center of mass of the disk is

$$<\mathbf{r}> = \frac{\sum M_j <\mathbf{r}_j>}{\sum M_j}\ , \qquad (17)$$

where

$$\begin{aligned} M_j <\mathbf{r}_j> &= \int d^2x\ \mathbf{r}\ \delta\Sigma_j\ ,\\ &= M_j\oint\frac{d\phi}{2\pi}\ \boldsymbol{\xi}_j\ , \end{aligned} \qquad (18)$$

where an integration by parts has been made.

We can write in general

$$\begin{aligned} \xi_{jr} &= \epsilon_{jx}\cos\phi + \epsilon_{jy}\sin\phi\ ,\\ \xi_{j\phi} &= -\delta_{jx}\sin\phi + \delta_{jy}\cos\phi\ . \end{aligned} \qquad (19)$$

Here, $\epsilon_{jx,y}$ and $\delta_{jx,y}$ are the *ring displacement amplitudes*: $\epsilon_{jx,y}$ represents the shift of the ring's center, and $\delta_{jx,y}$ represents in general both the shift of the ring's center *and* the azimuthal displacement of the ring matter. Firstly, notice that for $\epsilon_{jx,y} = 0$ and $\delta_{jx,y} \neq 0$, there is no shift of the ring's center but rather an azimuthal displacement of the ring matter. In this case $\delta\Sigma_j = -(\Sigma_j/r_j)(\partial\xi_{j\phi}/\partial\phi)$. Secondly, notice that $\delta_{jx,y} = \epsilon_{jx,y}$ corresponds to a *rigid* shift of the ring without azimuthal displacement of the ring matter. For example, a rigid shift in the $x-$direction has $\epsilon_{jx} = \delta_{jx}$ and $\epsilon_{jy} = 0 = \delta_{jy}$ so that $\xi_{jr} = \epsilon_{jx}\cos\phi$ and $\xi_{j\phi} = -\epsilon_{jx}\sin\phi$. In this case, $\nabla\cdot\boldsymbol{\xi}_j = 0$ so that $\delta\Sigma_j = -\nabla\cdot(\Sigma_j\boldsymbol{\xi}_j) = -\xi_{rj}(\partial\Sigma_j/\partial r)$. Figure 2 shows the nature of perturbations with a shift of the ring's center and with an azimuthal displacement of the ring matter.

Equation (18) now gives

$$\begin{aligned} <r_{jx}> &= \frac{1}{2}(\epsilon_{jx} + \delta_{jx})\ ,\\ <r_{jy}> &= \frac{1}{2}(\epsilon_{jy} + \delta_{jy})\ . \end{aligned} \qquad (20)$$

For the case of a rigid shift of a ring, $\epsilon_{jx,y} = \delta_{jx,y}$, the center of mass position is simply $<r_{jx,y}> = \epsilon_{jx,y}$, as expected.

Similarly, the velocity perturbation of the disk can be written as

$$<\delta\mathbf{v}> = \frac{\sum M_j <\delta\mathbf{v}_j>}{\sum M_j}\ , \qquad (21)$$

where

$$<\delta\mathbf{v}_j> = \frac{1}{M_j}\int d^2x\ \ (\delta\Sigma_j\mathbf{v} + \Sigma_j\delta\mathbf{v}_j)\ . \qquad (22)$$

Evaluation of (22) gives

$$\begin{aligned} <\delta v_{jx}> &= \frac{1}{2}\left(\dot\epsilon_{jx} + \dot\delta_{jx}\right),\\ <\delta v_{jy}> &= \frac{1}{2}\left(\dot\epsilon_{jy} + \dot\delta_{jy}\right)\ . \end{aligned} \qquad (23)$$

The influence of the eccentric motion on the rotation curves is discussed at the end of §6.3.

Fig. 2.— Drawing of two perturbed rings with equilibrium radii $r_1 = 1$ and $r_2 = 2$ in arbitrary units. The center of ring 1 is rigidly shifted in the $x-$direction from the origin by $\epsilon_{1x} = 0.4$ and $\epsilon_{1y} = 0$. The center of ring 2 is at the origin while the distribution of matter around the ring is perturbed as indicated by the small circles. For this ring, $\delta_{2x} = 0.4$ and $\delta_{2y} = 0$.

2.4. *Ring Equations of Motion*

Equations (12) and (19) give the ring equations of motion,

$$\begin{aligned}
\ddot{\epsilon}_x + 2\Omega\dot{\epsilon}_y - \tilde{\Omega}^2\epsilon_x - 2\Omega(\dot{\delta}_y - \Omega\delta_x) &= \ < \delta F_r^C > \ , \\
\ddot{\epsilon}_y - 2\Omega\dot{\epsilon}_x - \tilde{\Omega}^2\epsilon_y + 2\Omega(\dot{\delta}_x + \Omega\delta_y) &= \ < \delta F_r^S > \ , \\
\ddot{\delta}_x + 2\Omega\dot{\delta}_y - \Omega^2\delta_x - 2\Omega(\dot{\epsilon}_y - \Omega\epsilon_x) &= - < \delta F_\phi^S > \ , \\
\ddot{\delta}_y - 2\Omega\dot{\delta}_x - \Omega^2\delta_y + 2\Omega(\dot{\epsilon}_x + \Omega\epsilon_y) &= \ < \delta F_\phi^C > \ ,
\end{aligned} \tag{24}$$

where the j subscripts are implicit, where the angular brackets indicate the average over the ring $< (..) > \equiv 2\pi \int rdr(..)\Sigma_j(r)/M_j$, where $\tilde{\Omega}^2 \equiv \Omega^2 - 2\Omega r\Omega'$, and

$$\delta F_\alpha^{C,S} \equiv \oint \frac{d\phi}{\pi} \ [\cos\phi, \ \sin\phi] \, \delta F_\alpha \ ,$$

with $\alpha = r, \ \phi$.

We now evaluate the different force terms on the right-hand side of (24). For this it is useful to write $\delta\Sigma = \delta\Sigma_a + \delta\Sigma_b$, where

$$\delta\Sigma_a = -\frac{1}{r}\frac{\partial(r\Sigma\xi_r)}{\partial r} \ , \quad \delta\Sigma_b = -\frac{1}{r}\frac{\partial(\Sigma\xi_\phi)}{\partial\phi} \ , \tag{25}$$

from equation (13). The corresponding contributions to the potential, $\delta\Phi = \delta\Phi_a + \delta\Phi_b$ evaluated at (r, ϕ) are from (13),

$$\delta\Phi_a(r,\phi) = G\sum_k M_k(\epsilon_{kx}\cos\phi + \epsilon_{ky}\sin\phi) \ \times$$

$$\int_0^\infty r'dr' \left\{\frac{1}{r'}\frac{\partial}{\partial r'}\left[r'S(r'|r_k)\right]\right\} \oint \frac{d\Psi'}{2\pi} \ \frac{\cos\Psi}{R(r,r')} \ , \tag{26}$$

and

$$\delta\Phi_b(r,\phi) = -G\sum_k M_k(\delta_{kx}\cos\phi + \delta_{ky}\sin\phi) \ \times$$

$$\int_0^\infty r'dr' \ \frac{S(r'|r_k)}{r'} \oint \frac{d\Psi}{2\pi} \ \frac{\cos\Psi}{R(r,r')} \tag{27}$$

where $R^2(r,r') \equiv r^2 + (r')^2 - 2rr'\cos\Psi$,

$$S(r|r_k) \equiv \frac{2\pi\Sigma_k(r)}{M_k} = \frac{\exp[-(r-r_k)^2/2\Delta r_k^2]}{r\sqrt{2\pi}\Delta r_k} \ ,$$

and $\Psi \equiv \phi' - \phi$.

Evaluation of the force on the $j^{\rm th}$ ring due to the other rings gives

$$\begin{aligned}
M_j < \delta F_{rj}^C > &= \sum_k (C_{jk}\epsilon_{kx} + D_{jk}\delta_{kx}) \ , \\
M_j < \delta F_{rj}^S > &= \sum_k (C_{jk}\epsilon_{ky} + D_{jk}\delta_{ky}) \ , \\
-M_j < \delta F_{\phi j}^S > &= \sum_k \left(E_{jk}\delta_{kx} + D'_{jk}\epsilon_{kx}\right) \ , \\
M_j < \delta F_{\phi j}^C > &= \sum_k \left(E_{jk}\delta_{ky} + D'_{jk}\epsilon_{ky}\right) ,
\end{aligned} \tag{28}$$

where the 'tidal coefficients' are

$$C_{jk} = -GM_jM_k \iint rdr \ r'dr' \ \times$$

$$S(r|r_j)\frac{\partial\big[r'S(r'|r_k)\big]}{r'\ \partial r'}\frac{\partial\mathcal{K}(r,r')}{\partial r}\ , \qquad (29)$$

$$D_{jk} = GM_jM_k\iint rdr\ r'dr'\ \times$$

$$S(r|r_j)S(r'|r_k)\frac{\partial\mathcal{K}(r,r')}{r'\ \partial r} \qquad (30)$$

$$D'_{jk} = -GM_jM_k\iint rdr\ r'dr'\ \times$$

$$\frac{S(r|r_j)}{r}\frac{\partial\big[r'S(r'|r_k)\big]}{r'\partial r'}\ \mathcal{K}(r,r')\ , \qquad (31)$$

$$E_{jk} = GM_jM_k\iint rdr\ r'dr'\ \times$$

$$S(r|r_j)S(r'|r_k)\ \frac{\mathcal{K}(r,r')}{r\ r'}\ , \qquad (32)$$

where

$$\mathcal{K}(r,r') \equiv \oint \frac{d\Psi}{2\pi}\frac{\cos\Psi}{R(r,r')}\ , \qquad (33)$$

and where the r, r' integrals are all from 0 to ∞.

Formal integration by parts of (29) gives

$$C_{jk} = GM_jM_k\iint rdr\ r'dr'\ \times$$

$$S(r|r_j)S(r'|r_k)\frac{\partial^2\mathcal{K}(r,r')}{\partial r'\ \partial r}\ , \qquad (34)$$

which shows that $C_{jk} = C_{kj}$. Also, integration by parts of (31) gives

$$D'_{jk} = GM_jM_k\iint rdr\ r'dr'\ \times$$

$$S(r|r_j)S(r'|r_k)\ \frac{\partial\mathcal{K}(r,r')}{r\ \partial r'}\ , \qquad (35)$$

and this shows that $D_{jk} = D'_{kj}$. Expressions for the tidal coefficients in terms of elliptic integrals are given in the Appendix.

Following the approach of Lovelace (1998), we introduce the complex displacement amplitudes

$$\mathcal{E}_j \equiv \epsilon_{jx} - i\epsilon_{jy} = \epsilon_j(t)\exp[-i\varphi_j(t)]\ , \qquad (36)$$

$$\Delta_j \equiv \delta_{jx} - i\delta_{jy} = \delta_j(t)\exp[-i\psi_j(t)]\ . \qquad (37)$$

Here, $\epsilon_j \geq 0$ is the amplitude of the shift of the ring's center, and φ_j is angle of the shift with respect to the $x-$axis; $\delta_j \geq 0$ is the amplitude of the azimuthal displacement of the ring matter, and ψ_j is the angle of the maximum of the ring density also with respect to the $x-$axis. If $\varphi(t)$ and $\psi(t)$ increase with time, the ring precesses in the same sense as the particle motion and we refer to this as *forward precession.* The opposite case, with $\varphi(t)$ and $\psi(t)$ decreasing with time, is termed *backward precession.*

We now combine equations (24) and (28) to obtain the ring equations of motion,

$$M_j\left(\ \ddot{\mathcal{E}}_j + 2i\Omega_j\dot{\mathcal{E}}_j - \tilde{\Omega}_j^2\mathcal{E}_j - 2i\Omega_j\dot{\Delta}_j + 2\Omega_j^2\Delta_j\right) = \sum_k\left(C_{jk}\mathcal{E}_k + D_{jk}\Delta_k\right)\ , \qquad (38)$$

$$M_j\left(\ddot{\Delta}_j + 2i\Omega_j\dot{\Delta}_j - \Omega_j^2\Delta_j - 2i\Omega_j\dot{\mathcal{E}}_j + 2\Omega_j^2\mathcal{E}_j\right) = \sum_k\left(E_{jk}\Delta_k + D'_{jk}\mathcal{E}_k\right)\ , \qquad (39)$$

where $j = 1..N$, $\Omega_j \equiv \Omega(r_j)$, and $\tilde{\Omega}_j \equiv \tilde{\Omega}(r_j)$.

2.5. *Renormalization of Ring Equations*

Here, we redo the ring equations of motion so as to diminish the strong tidal interactions of nearest neighbor rings due to the terms $\propto C_{j,j+k}$. First, we rewrite the right-hand side of (38) as

$$\sum_k\left[C_{jk}\left(\mathcal{E}_k - \mathcal{E}_j\right) + D_{jk}\left(\Delta_k - \mathcal{E}_j\right)\right] + \mathcal{E}_j\sum_k(C_{jk} + D_{jk})\ , \qquad (40)$$

Similarly, we rewrite the right-hand side of (39) as

$$\sum_k\left[E_{jk}\left(\Delta_k - \Delta_j\right) + D'_{jk}\left(\mathcal{E}_k - \Delta_j\right)\right] + \Delta_j\sum_k\left(E_{jk} + D'_{jk}\right)\ . \qquad (41)$$

We define

$$\Omega_{\mathcal{E}j}^2 \equiv \tilde{\Omega}_j^2 + \frac{1}{M_j}\sum_k\left(C_{jk} + D_{jk}\right)\ . \qquad (42)$$

Using the relations $\tilde{\Omega}^2 = \Omega^2 - r(d\Omega^2/dr)$, $\Omega^2 = \Omega_d^2 + \Omega_b^2 + \Omega_h^2$, and equation (A14) of the Appendix gives

$$\Omega_{\mathcal{E}j}^2 = \Omega_j^2 - \left(r\frac{\partial(\Omega_b^2 + \Omega_h^2)}{\partial r}\right)_j + \frac{1}{M_j}\sum_k(D'_{jk} + E_{jk})\ . \qquad (43)$$

(If there is a central black hole M_{bh} then the right-hand side of (43) also has the term $-[r(\partial\Omega_{bh}^2/\partial r)]_j = 3GM_{bh}/r_j^3$.) Similarly, we define

$$\Omega_{\Delta j}^2 \equiv \Omega_j^2 + \frac{1}{M_j}\sum_k(D'_{jk} + E_{jk})\ . \qquad (44)$$

The ring equations of motion now become

$$M_j\left(\ \ddot{\mathcal{E}}_j + 2i\Omega_j\dot{\mathcal{E}}_j - \Omega_{\mathcal{E}j}^2\mathcal{E}_j - 2i\Omega_j\dot{\Delta}_j + 2\Omega_j^2\Delta_j\right) = \sum_k\left[C_{jk}\left(\mathcal{E}_k - \mathcal{E}_j\right) + D_{jk}\left(\Delta_k - \mathcal{E}_j\right)\right]\ , \qquad (45)$$

$$M_j\left(\ddot{\Delta}_j + 2i\Omega_j\dot{\Delta}_j - \Omega_{\Delta j}^2\Delta_j - 2i\Omega_j\dot{\mathcal{E}}_j + 2\Omega_j^2\mathcal{E}_j\right) = \sum_k\left[E_{jk}\left(\Delta_k - \Delta_j\right) + D'_{jk}\left(\mathcal{E}_k - \Delta_j\right)\right]\ . \qquad (46)$$

In the large N limit, the sums in equations (45) and (46) go over to bounded integrals. The diagonal elements, C_{jj} (and E_{jj}), are absent from (45) and (46). Note also that the self-interaction of a ring vanishes as it should for the case of a rigid shift where $\mathcal{E}_j = \Delta_j$.

2.6. *Dynamics and Influence of "Center"*

The dynamical equations (45) and (46) do not account for the part of the disk inside the innermost ring r_1. Also, there may be a massive black hole M_{bh} near the galaxy center. We treat this central region separately as a point mass M_0,

$$M_0 = M_{bh} + 2\pi \int_0^{r_1'} rdr\ \Sigma_d(r) \ , \qquad (47)$$

where $r_1' \equiv r_1 - \delta r_1/2$.

The horizontal displacement of the "center" is

$$\boldsymbol{\epsilon}_0(t) = \epsilon_{0x}\hat{\mathbf{x}} + \epsilon_{0y}\hat{\mathbf{y}} \ . \qquad (48)$$

The equation of motion for $\boldsymbol{\epsilon}_0$ taking into account the eccentric displacemnts of the rings is

$$M_0\left(\frac{d^2\mathcal{E}_0}{dt^2} + \Omega_0^2\mathcal{E}_0\right) = -\sum_{k=1}^{N} F_{0k}\left(\mathcal{E}_k - \frac{1}{2}\Delta_k\right) \ , \qquad (49)$$

where

$$\mathcal{E}_0 \equiv \epsilon_{0x} - i\epsilon_{0y} = \epsilon_0 \exp(-i\varphi_0) \qquad (50)$$

is the complex displacement amplitude of the "center"; $F_{0k} \equiv GM_0M_k/r_k^3$, $k = 1..N$, are the tidal coefficients between the "center" and the rings; and

$$\Omega_0^2 = \left(\frac{1}{r}\frac{\partial\Phi}{\partial r}\right)_0 = \left(\frac{1}{r}\frac{\partial(\Phi_b+\Phi_h)}{\partial r}\right)_0 - \frac{1}{2M_0}\sum_j F_{0j} \qquad (51)$$

is the angular oscillation frequency of a particle at the galaxy center. From §2.1, we have $[(1/r)(\partial\Phi_b/\partial r)]_0 = GM_b/r_b^3$ and $[(1/r)(\partial\Phi_h/\partial r)]_0 = v_h^2/r_h^2$. For all of the considered conditions, we find $\Omega_0^2 > 0$.

The influence of the displaced "center" on the rings is included by adding the force terms, due to the "center's" displacement, to the right-hand sides of equations (45) and (46),

$$M_j\left(\ \ddot{\mathcal{E}}_j\ +\ ..\right) = -\ 2F_{0j}\mathcal{E}_0\ +\ ..\ , \qquad (52)$$

$$M_j\left(\ \ddot{\Delta}_j\ +\ ..\right) = +\ F_{0j}\mathcal{E}_0\ +\ ..\ , \qquad (53)$$

where the ellipses denote terms in equations (45) and (46).

2.7. *Energy Conservation*

An energy constant of the motion of the ring system can be obtained by multiplying equation (52) by $\dot{\mathcal{E}}_j^*$ and (53) by $\dot{\Delta}_j^*$ (with $[...]^*$ denoting the complex conjugate), adding the two equations, summing over j, and dividing by 2. (This factor of 2 makes the kinetic energy of a rigidly shifted ring with $\boldsymbol{\epsilon}_j = \boldsymbol{\delta}_j$ equal to $(1/2)M_j\dot{\boldsymbol{\epsilon}}_j^2$.) Further, we multiply equation (49) by $\mathcal{E}_0^*$ and add the result to the previous sum. In this way we find $dE/dt = 0$, where

$$E = \frac{1}{4}\sum_j M_j\left(\dot{\boldsymbol{\epsilon}}_j^2 - \Omega_{\mathcal{E}j}^2\boldsymbol{\epsilon}_j^2 + \dot{\boldsymbol{\delta}}_j^2 - \Omega_{\Delta j}^2\boldsymbol{\delta}_j^2 + 4\Omega_j^2\boldsymbol{\epsilon}_j\cdot\boldsymbol{\delta}_j\right)$$
$$+\frac{1}{8}\sum_{k\neq j}\sum_j\Big[C_{jk}(\boldsymbol{\epsilon}_j-\boldsymbol{\epsilon}_k)^2 + 2D_{jk}(\boldsymbol{\epsilon}_j-\boldsymbol{\delta}_k)^2 +$$
$$+E_{jk}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_k)^2\Big]$$
$$+\frac{1}{2}M_0(\dot{\boldsymbol{\epsilon}}_0^2 + \Omega_0^2\boldsymbol{\epsilon}_0^2) + \sum_j F_{0j}\left(\boldsymbol{\epsilon}_0\cdot\boldsymbol{\epsilon}_j - \frac{1}{2}\boldsymbol{\epsilon}_0\cdot\boldsymbol{\delta}_j\right) \ , \qquad (54)$$

and where the real vectors $\boldsymbol{\epsilon} = \epsilon_x\hat{\mathbf{x}} + \epsilon_y\hat{\mathbf{y}}$ and $\boldsymbol{\delta} = \delta_x\hat{\mathbf{x}} + \delta_y\hat{\mathbf{y}}$ are useful here.

2.8. *Lagrangian*

By inspection, we find the Lagrangian for the ring system $\mathcal{L}(\epsilon_{jx}, \dot{\epsilon}_{jx}, ..)$,

$$\mathcal{L} = \frac{1}{4}\sum_j M_j\Big\{\dot{\boldsymbol{\epsilon}}_j^2 + \Omega_{\mathcal{E}j}^2\boldsymbol{\epsilon}_j^2 + \dot{\boldsymbol{\delta}}_j^2 + \Omega_{\Delta j}^2\boldsymbol{\delta}_j^2 - 4\Omega_j^2\boldsymbol{\epsilon}_j\cdot\boldsymbol{\delta}_j$$
$$-2\Omega_j[(\boldsymbol{\epsilon}_j-\boldsymbol{\delta}_j)\times(\dot{\boldsymbol{\epsilon}}_j-\dot{\boldsymbol{\delta}}_j)]\cdot\hat{\mathbf{z}}\Big\} + \frac{1}{2}M_0(\dot{\boldsymbol{\epsilon}}_0^2 - \Omega_0^2\boldsymbol{\epsilon}_0^2)$$
$$-\frac{1}{8}\sum_{k\neq j}\sum_j\Big[C_{jk}(\boldsymbol{\epsilon}_j-\boldsymbol{\epsilon}_k)^2 + 2D_{jk}(\boldsymbol{\epsilon}_j-\boldsymbol{\delta}_k)^2 +$$
$$+E_{jk}(\boldsymbol{\delta}_j-\boldsymbol{\delta}_k)^2\Big] - \sum_j F_{0j}\left(\boldsymbol{\epsilon}_0\cdot\boldsymbol{\epsilon}_j - \frac{1}{2}\boldsymbol{\epsilon}_0\cdot\boldsymbol{\delta}_j\right) \ , \qquad (55)$$

Because $\partial\mathcal{L}/\partial t = 0$, the Hamiltonian,

$$\mathcal{H} \equiv \sum_j\left(\dot{\epsilon}_{jx}\frac{\partial\mathcal{L}}{\partial\dot{\epsilon}_{jx}} + ...\right) - \mathcal{L} \ , \qquad (56)$$

is a constant of the motion. It is readily verified that $\mathcal{H} = E$.

We can make a canonical transformation $(\epsilon_{jx}, \epsilon_{jy}) \to (\epsilon_j, \varphi_j)$, $(\delta_{jx}, \delta_{jy}) \to (\delta_j, \psi_j)$ to obtain the Lagrangian as $\mathcal{L} = \mathcal{L}(\dot{\epsilon}_j, \dot{\delta}_j, \epsilon_j, \delta_j, \varphi_j, \psi_j)$. Note for example that $\dot{\boldsymbol{\epsilon}}_j^2 \to \dot{\epsilon}_j^2 + \epsilon_j^2\dot{\varphi}_j^2$. It is then clear from the azimuthal symmetry of the equilibrium that $\mathcal{L}$ is invariant under the simultaneous changes $\varphi_j \to \varphi_j + \theta$, $\psi_j \to \psi_j + \theta$ for $j = 1, .., N$, where θ is an arbitrary angle. Thus

$$\sum_j\left(\frac{\partial\mathcal{L}}{\partial\varphi_j} + \frac{\partial\mathcal{L}}{\partial\psi_j}\right) = 0 \ ,$$

and consequently the total canonical angular momentum of the ring system,

$$\mathcal{P}_\phi \equiv \sum_j\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}_j} + \frac{\partial\mathcal{L}}{\partial\dot{\psi}_j}\right) \ , \qquad (57)$$

is another constant of the motion. Evaluating (57) gives

$$\mathcal{P}_\phi = \frac{1}{2}\sum_j M_j\Big[\epsilon_j^2(\dot{\varphi}_j - \Omega_j) + \delta_j^2(\dot{\psi}_j - \Omega_j)$$
$$+2\Omega_j\epsilon_j\delta_j\cos(\varphi_j - \psi_j)\Big] + M_0\epsilon_0^2\dot{\varphi}_0 \ , \qquad (58)$$

where the last term represents the angular momentum of the galaxy center. The last term within the square brackets can also be written as $2\Omega_j(\boldsymbol{\epsilon}_j\cdot\boldsymbol{\delta}_j)$.

The constants of the motion $\mathcal{H}$ and $\mathcal{P}_\phi$ are valuable for checking numerical integrations of the equations of motion (52) and (53).

FIG. 3.— The top panel (**a**) of the figure shows the radial dependence of the four frequencies ω_α ($\alpha = 1,..,4$) of the modes of oscillation of an eccentric ring normalized by the disk's angular rotation rate $\Omega(r)$. The bottom panel (**b**) shows the corresponding radial dependence of the mode amplitude ratio $\mathcal{E}/\Delta$ as discussed in the text. The case shown is for the galaxy parameters of Figure 1 with innermost ring of radius $r_1 = 1$ kpc. The values Ω_j, $\Omega_{\mathcal{E}j}$, and $\Omega_{\Delta j}$ were obtained using equations (A11), (43), and (44).

3. ECCENTRIC MOTION OF A SINGLE RING

Consider the eccentric motion of a particular ring with the other rings not excited. This is *not* a self-consistent limit because gravitational interactions will in general excite all of the rings. However this limit is informative. With $\mathcal{E}_j \propto \Delta_j \propto \exp(-i\omega t)$, $j = 1,..,N$, where ω the ring precession frequency, we get

$$\left[(\omega-\Omega)^2 + E\right]\mathcal{E} + \left[2\Omega(\omega-\Omega) + d\,\right]\Delta = 0~,$$

$$\left[2\Omega(\omega-\Omega) + d\,\right]\mathcal{E} + \left[(\omega-\Omega)^2 + D\right]\Delta = 0~, \qquad (59)$$

where the j subscripts are implicit, $E \equiv \Omega^2_{\mathcal{E}j} - \Omega^2_j - D_{jj}/M_j$, $D \equiv \Omega^2_{\Delta j} - \Omega^2_j - D_{jj}/M_j$, and $d \equiv D_{jj}/M_j$. With $w \equiv \omega - \Omega$, we get

$$w^4 + (D + E - 4\Omega^2)w^2 - 4d\Omega w + ED - d^2 = 0~. \qquad (60)$$

For the limit of many rings, the terms involving d become negligible, and (60) can be readily solved to give four real roots if $0 < ED < (4\Omega^2 - D - E)^2/4$.

Figure 3a shows the radial dependence of the four ring precession frequencies ω_α ($\alpha = 1,..,4$) corresponding to the four modes of oscillation. These modes are the analogues of the normal modes of vibration of a non-rotating system (see Lovelace 1998).

Three of the modes in Figure 3a have positive frequencies ($\omega_\alpha > 0$ for $\alpha = 2,3,4$) so that they have forward precession with

$$\varphi_\alpha = \omega_\alpha t + \mathrm{const}~, \quad \text{and} \quad \psi_\alpha = \omega_\alpha t + \mathrm{const}'~. \qquad (61)$$

These relations follow from (36) and (37) because $|\mathcal{E}| = \epsilon$ and $|\Delta| = \delta$ are constants. The other mode ($\alpha = 1$) has $\omega_1 < 0$ and therefore backward precession.

Figure 3b shows that the two modes $\alpha = 2,3$ have $\mathcal{E}/\Delta$ small compared with unity near the inner radius of the disk, r_1. Note that $\mathcal{E}/\Delta = 0$ corresponds to a pure azimuthal shift of the ring matter without a shift of the ring center. The mode $\alpha = 1$ with backward precession has $\mathcal{E}/\Delta \sim 1$. As mentioned, $\mathcal{E}/\Delta = 1$ corresponds to a rigid shift of the ring center.

Evaluation of the ring energy for the four modes using equation (54) shows that the modes $\alpha = 1,2$ have *negative energy* whereas the modes $\alpha = 3,4$ have positive energy. The negative energy modes are unstable in the presence of dissipation, for example, the force due to dynamical friction (see for example Lovelace 1998).

Note that for vanishing ring mass ($D \propto d \propto M_j \to 0$), the middle two roots approach $\omega = \Omega \pm [DE(4\Omega^2 - E)]^{1/2}$. Thus for $M_j \to 0$, there are only three different roots of equation (60), $\omega = \Omega$ and $\omega = \Omega \pm (4\Omega^2 - E)^{1/2}$.

Outside of the central region of a galaxy we have $\Omega \sim 1/r$. Consequently, the radial dependences of the mode frequencies $\omega_\alpha(r) \propto \pm 1/r$ will tend to "wrap up" an initially coherent asymmetry into a tightly wrapped spiral (in the absence of ring interactions). The forward precessing modes ($\alpha = 2-4$) will give a *trailing* spiral wave, $\varphi_\alpha \propto 1/r$, (with respect to the azimuthal motion $v_\phi > 0$), whereas the backward precessing mode ($\alpha = 1$) will give a *leading* spiral wave, $\varphi_1 \propto -1/r$. The case of mode $\alpha = 1$ for non-interacting rings was discussed earlier by Baldwin *et al.* (1980).

4. ECCENTRIC MOTION OF A "DISK" OF TWO RINGS

Here, we consider the eccentric motion of a "disk" consisting of two interacting rings, one of mass M_1 and radius r_1, and the other of mass M_2, radius r_2. The values of Ω^2_j, $\Omega^2_{\mathcal{E}j}$, $\Omega^2_{\Delta j}$ ($j = 1,2$) are given by equations (A11), (43), and (44) with the bulge and halo potentials as given in §2.1. Thus the present treatment is *self-consistent* (in contrast with the previous subsection). With $\mathcal{E}_j$ and Δ_j proportional to $\exp(-i\omega t)$, equations (45) and (46) give

$$\begin{aligned}\big[-(\omega-\Omega_1)^2+\Omega_1^2-\Omega_{\mathcal{E}1}^2+D_{11}\big]\mathcal{E}_1 \\ +\big[2\Omega_1(\Omega_1-\omega)-D_{11}\big]\Delta_1 \\ =\big[C_{12}(\mathcal{E}_2-\mathcal{E}_1)+D_{12}(\Delta_2-\mathcal{E}_1)\big]/M_1\ ,\end{aligned} \qquad (62)$$

$$\begin{aligned}\big[-(\omega-\Omega_1)^2+\Omega_1^2-\Omega_{\Delta 1}^2+D'_{11}\big]\Delta_1 \\ +\big[2\Omega_1(\Omega_1-\omega)-D'_{11}\big]\mathcal{E}_1 \\ =\big[E_{12}(\Delta_2-\Delta_1)+D'_{12}(\mathcal{E}_2-\Delta_1)\big]/M_1\ ,\end{aligned} \qquad (63)$$

$$\begin{aligned}\big[-(\omega-\Omega_2)^2+\Omega_2^2-\Omega_{\mathcal{E}2}^2+D_{22}\big]\mathcal{E}_2 \\ +\big[2\Omega_2(\Omega_2-\omega)-D_{22}\big]\Delta_2 \\ =\big[C_{12}(\mathcal{E}_1-\mathcal{E}_2)+D'_{12}(\Delta_1-\Delta_2)\big]/M_2\ ,\end{aligned} \qquad (64)$$

$$\begin{aligned}\big[-(\omega-\Omega_2)^2+\Omega_2^2-\Omega_{\Delta 2}^2+D'_{22}\big]\Delta_2 \\ +\big[2\Omega_2(\Omega_2-\omega)\mathcal{E}_2-D'_{22}\big] \\ =\big[E_{12}(\Delta_1-\Delta_2)+D_{12}(\mathcal{E}_1-\Delta_2)\big]/M_2\ .\end{aligned} \qquad (65)$$

For a non-zero solution, the determinant of the 4×4 matrix multiplying ($\mathcal{E}_1,\ \Delta_1,\ \mathcal{E}_2,\ \Delta_2$) must be zero. This leads to an eighth order polynomial in ω which can be readily solved (with Maple R. 5) for the frequencies of the 8 modes ($\alpha = 1..8$).

Figure 4 shows the behavior, including instability, of a system of two rings of equal mass $M_1 = M_2$ so that the "disk" mass is $M_d = 2M_1$. For $M_d \to 0$, the modes $\alpha = 3, 4$ approach Ω_2 and the modes $\alpha = 5, 6$ approach Ω_1 which agrees with the behavior found in §3. Note that for small M_d modes $\alpha = 1-4$ are associated with ring 2 while modes $\alpha = 5-8$ are associated with ring 1. In the absence of interactions, the rings are stable independent of their mass. As the disk mass increases, the modes $\alpha = 4, 5$ approach each other and merge at $M_d \approx 0.276\times 10^{10}M\odot$ and give instability with $\mathcal{I}m(\omega) \equiv \omega_i > 0$ as shown in Figure 4b. We refer to this as the "first instability." Notice that the onset of instability corresponds to a merging of the positive energy mode $\alpha = 4$ of ring 2 with the negative energy mode $\alpha = 5$ of ring 1 (see §3). The interaction of positive and negative energy modes is a well-known instability mechanism (see for example Lovelace, Jore, & Haynes 1997). At the instability threshold, a dimensionless measure of the ring self-gravity is $GM_d/(\bar{r}^3\bar{\Omega}^2) \sim 0.1$, where $\bar{r} = 4$ kpc and $\bar{\Omega} \approx 0.0458/t_o$. The dependence of the growth rate is well fitted by $\omega_i t_o \approx 0.00842(M_d - M_{c1})^{0.67}$, with the masses in units of $10^{10}M_\odot$ and $t_o \equiv 10^6$ yr. For $M_d > 3.12\times 10^{10} \equiv M_{c2}$, there is a "second instability" with growth rate $\omega_i t_o \approx 0.0196(M_d - M_{c2})^{0.55}$.

The ratios of the complex perturbation amplitudes can readily be obtained from (62) - (65) once the 8, possibly complex frequencies are known. For the "first instability," for example, for $M_d = 10^{10}M_\odot$ and the same conditions as for Figure 4, we find $\mathcal{E}_1 \approx 0.254 - 0.117i$, $\Delta_1 = 1$ (by choice), $\mathcal{E}_2 \approx 0.0108 - 0.365i$, and $\Delta_2 \approx 0.673 + 1.30i$. These values correspond to $\varphi_1 \approx 24.7°$, $\varphi_2 \approx 91.7°$, $\psi_1 = 0$ (by choice), and $\psi_2 \approx 297°$. Thus the azimuthal density enhancement in the outer ring *trails* the density enhancement of the inner ring. On the other hand, the radial shift of the outer ring *leads* the shift of the inner ring.

As a second example, for $M_d = 4\times 10^{10}M_\odot$ both the "first" and "second" instability occur. For the "first" instability, we again find that the azimuthal density enhancement of the outer ring *trails* that of the inner ring, whereas the radial shift of the outer ring *leads* that of the inner ring. For the "second" instability the situation is different in that *both* the azimuthal density enhancement and the radial shift of the outer ring *trail* those in the inner ring. Specifically, we find $\varphi_1 \approx 178°$, $\varphi_2 \approx 89.3°$, $\psi_1 = 0$ (by choice), and $\psi_2 \approx 283°$. Note that for both rings, the angle of the radial shift is roughly 180° displaced from the azimuthal density enhancement. The displacement of the center of mass of the ring (equation 20) is dominated by the azimuthal displacement ($|\mathcal{E}_1| \approx 0.70|\Delta_1|$ and $|\mathcal{E}_2| \approx 0.67|\Delta_2| \approx 0.22|\Delta_1|$). Thus having φ_j and ψ_j about 180° out of phase allows the center of mass of each ring to be closer to the origin, and this is a lower energy configuration.

Consider now the angular momentum of the perturbed two ring system which from (58) is $\mathcal{P}_\phi = \mathcal{P}_{1\phi} + \mathcal{P}_{2\phi} =$ const. For the evaluation of $\mathcal{P}_\phi$, note that for each of the eight modes, $\dot{\Delta}_j = -i\omega_\alpha\Delta_j$, and $\dot{\mathcal{E}}_j = -i\omega_\alpha\mathcal{E}_j$, where $\alpha = 1, .., 8$ labels the mode. This implies that

$$\omega_\alpha = \dot{\psi}_j + i(\dot{\delta}_j/\delta_j)\ =\ \dot{\varphi}_j + i(\dot{\epsilon}_j/\epsilon_j)\ . \qquad (66)$$

Thus, for the "first instability," for $M_d = 10^{10}M_\odot$ where $\omega t_o \approx 0.0500 + 0.00719i$, we have $\dot{\psi}_j = \dot{\varphi}_j = 0.05/t_o$, (which corresponds to forward precession), and $\dot{\delta}_j/\delta_j = \dot{\epsilon}_j/\epsilon_j = 0.00719/t_o$. For an unstable mode, the coefficients of the six terms in $\mathcal{P}_\phi$ are all grow exponentially. The only possible way in which $\mathcal{P}_\phi$ =const can be maintained is to have $\mathcal{P}_\phi = 0 = \mathcal{P}_{\phi 1} + \mathcal{P}_{\phi 2}$. This relation provides a useful check on the correctness of the calculations.

For the "first instability" ($M_d > M_{c1}$), we find by evaluating (58) that $\mathcal{P}_{\phi 1} = -\mathcal{P}_{\phi 2} > 0$. This means that the angular momentum of the inner ring increases while that of the outer ring decreases. Thus the instability acts to transfer angular momentum *inward*. In contrast, for the "second instability" ($M_d > M_{c2}$), we find that $\mathcal{P}_{\phi 1} = -\mathcal{P}_{\phi 2} < 0$. Thus, the "second instability" acts to transfer angular momentum *outward*. If $P_{\phi 1}$ decreases and $P_{\phi 2}$ increases, then the average radius of ring 1 must decrease and that of ring 2 must increase. Therefore, the "second instability" may be important for the accretion of matter in a gravitating disk.

5. ECCENTRIC MOTION OF ONE RING AND "CENTER"

Here, we consider the eccentric motion of a "disk" of one ring including the influence of the eccentric motion of the "center" M_0 which is located at $\mathbf{r} = 0$ in equilibrium. The mass M_0 includes the mass of a central black hole M_{bh} if it is present. The ring perturbation is described by $\mathcal{E}(t)$ and $\Delta(t)$ as given by (52) and (53), while the "center" is described by $\mathcal{E}_0(t)$ which is given by (49). With the perturbations $\propto \exp(-i\omega t)$, we find

$$\big[(\omega-\Omega)^2+E\big]\,\mathcal{E}+\big[2\Omega(\omega-\Omega)+d\ \big]\,\Delta = 2\Omega_a^2\mathcal{E}_0\ ,$$

FIG. 4.— The figure shows the eccentric instability of a "disk" of mass M_d consisting of two rings of equal mass $M_d/2$ at equilibrium radii $r_1 = 3$ and $r_2 = 5$ kpc. The top panel (**a**) shows the dependence of the real parts of the frequencies $\omega_{r\alpha}$ ($\alpha = 1, .., 8$) on M_d. The labels $u1, u2$ indicate unstable branches where the frequency is complex; $u1$ is referred to in the text as the "first instability" and $u2$ the "second instability." Here, $t_o \equiv 10^6$ yr. The bulge has $M_b = 0.5 \times 10^{10} M_\odot$ and $r_b = 1$ kpc, and the halo $v_h = 250$ km/s and $r_h = 5$ kpc in the expressions given in §2.1. The bottom panel (**b**) shows the dependence of the growth rate ω_i on the ring mass. The onset of instability corresponds to the merging of the two real frequencies in panel (**a**). The tidal coefficients, $\{C_{jk}\}$,, etc., are obtained using the equations of the Appendix and $\Delta r_j = 2$ kpc.

$$[2\Omega(\omega-\Omega)+d\,]\,\mathcal{E}+\left[(\omega-\Omega)^2+D\right]\Delta = \Omega_a^2\mathcal{E}_0 \ ,$$
$$(\omega^2-\Omega_0^2)\mathcal{E}_0 = \Omega_b^2\left(\mathcal{E}-\frac{\Delta}{2}\right) \ , \qquad (67)$$

where $\Omega_a^2 \equiv GM_0/r_1^3$ and $\Omega_b^2 \equiv GM_1/r_1^3$, with M_1 and r_1 the mass and radius of the ring. For a non-zero solution, the determinant of the 3×3 matrix multiplying $(\mathcal{E}, \Delta, \mathcal{E}_0)$ must be zero. This leads to a sixth order polynomial in ω or $w \equiv \omega - \Omega$ which can readily be solved (with Maple, R5) for the frequencies of the 6 modes ω_α ($\alpha = 1..6$). We obtain

$$\left[(w+\Omega)^2-{\Omega_0}^2\right]\left[\left(w^2+E\right)\left(w^2+D\right)-(2\Omega w+d)^2\right]$$
$$-\Omega_{ab}^4\left(\frac{5w^2}{2}+2D+\frac{E}{2}+2\Omega w+d\right)=0 \ , \qquad (68)$$

where the strength of the interaction between the ring and the "center" is measured by

$$\Omega_{ab}^2 \equiv \Omega_a\Omega_b = \frac{G\sqrt{M_0M_1}}{r_1^3} \ . \qquad (69)$$

Here, $D,\ E,\ d$ and $\Omega = \Omega_1$ are defined in (59), and Ω_0 is defined in (51).

Figure 5 shows the dependence of the growth rate $\omega_i = \mathcal{I}m(\omega_\alpha)$ on the mass of the "center" M_0 with the mass of the ring held fixed. The associated real part of the frequency is positive. The onset of instability corresponds to the point where two of the six modes with frequencies given by (68) merge. The merging is again of positive and negative energy modes. The growth rate has to a good approximation the dependence $\omega_i t_o \approx 0.00764(M_0 - 2.61)^{1/2}$, with M_0 in units of $10^9 M_\odot$ and $t_o = 10^6$ yr.

The ratios of the complex amplitudes follow from (67), and for the unstable mode for $M_0 = 2.75 \times 10^9 M_\odot$ we find $\omega t_o \approx 0.155 + 0.00289i$, $\Omega t_o \approx 0.131$, $\Omega_0 t_o \approx 0.142$, $\Omega_{ab} \approx 0.0465$, $\mathcal{E} \approx -0.817 + 0.0333i$, $\Delta = 1$ (by choice), and $\mathcal{E}_0 \approx -3.38 + 0.916i$ which give $|\mathcal{E}| \approx 0.817$, $|\mathcal{E}_1| \approx 3.51$, $\varphi \approx 182°$, and $\varphi_0 \approx 195°$, where $t_o = 10^6$ yr. Thus, the radial shift of the ring is roughly 180° away from the maximum of the density enhancement which has $\psi = 0$ (by choice). The shift of the center of mass of the ring (equation 20) is dominated by the azimuthal density enhancement. Thus the radial shift and azimuthal displacements are such that the center of mass of the ring moves closer to the origin which is a lower energy configuration. Note that the radial shift of the "center" *trails* the ring center of mass by an angle $360° - \varphi_0 \approx 165°$. Hence, the torque of the "center" on the ring acts to *reduce* the angular momentum of the ring as verified below. At the same time the ring acts to increase the angular momentum of the "center."

Consider now the angular momentum of the perturbed ring plus "center" system which is given by (58). Following the arguments of the prior section, the sum of the angular momentum of the "center" and that of the ring

FIG. 5.— The figure shows the eccentric instability of a ring of mass M_1 interacting gravitationally with a displaced point mass M_0 shifted from its equilibrium position $\mathbf{r} = 0$. The halo and bulge potentials are the same as in Figure 1. The mass M_0 includes the mass of a central black hole M_{bh} if present. The mass of the ring is $M_1 = 6 \times 10^{10} M_\odot$ and its radius is $r_1 = 3$ kpc. Here, $t_o \equiv 10^6$ yr. The tidal coefficients are evaluated using the expressions of the Appendix with $\Delta r_1 = 2$ kpc.

must be zero for a growing mode. The angular momentum of the "center" is simply $M_0 \epsilon_0^2 \dot{\varphi}_0$. For a pure mode, we have $\mathcal{R}e(\omega_\alpha) = \dot{\varphi}_0$. As mentioned, the real part of the frequency is positive for the unstable mode and therefore the angular momentum of the "center" increases while the angular momentum of the ring decreases. Thus, there is a transfer of angular momentum from the ring to the "center." Due to the loss of angular momentum the average radius of the ring will decrease. Thus the instability may be important for accretion of matter to the galaxy center.

6. ECCENTRIC MOTION OF N RINGS

For the results presented here, the rings are taken to be uniformly spaced in r with radii $r_j = 1 + (j-1)\delta r$ kpc with $\delta r = 0.5$ kpc and with $j = 1, .., N = 31$. The value $N = 31$ gives good spatial resolution over all but the inner part of the disk. The outer radius r_N (in the range say $10-20$ kpc) has little influence on the eccentric motion described here, as verified by comparing results with $r_N = 16$ kpc those obtained with significantly larger r_N. Also, the eccentric motion of the outer disk, say, $r \gtrsim 3$ kpc, is essentially independent of δr. We first consider in §6.1 the case where the ring masses correspond to the exponential distribution discussed in §2.1. The inner part of the disk is found to be strongly unstable to eccentric motions and therefore in §6.2 we consider a disk with the mass of the innermost three rings reduced. In §6.3 we consider disks with a smooth reduction in $\Sigma_d(r)$ in the inner part of the disk, $r \lesssim r_d$.

We solve (52) and (53) numerically as eight first order equations for ϵ_{xj}, $\dot{\epsilon}_{xj}$, ϵ_{yj}, and $\dot{\epsilon}_{yj}$, and for δ_{xj}, $\dot{\delta}_{xj}$, δ_{yj}, and $\dot{\delta}_{yj}$, $j = 1, .., N$. At the same time, we solve the two additional equations,

$$\frac{d\varphi_j}{dt} = \frac{\epsilon_{xj}\dot{\epsilon}_{yj} - \epsilon_{yj}\dot{\epsilon}_{xj}}{\epsilon_{xj}^2 + \epsilon_{yj}^2} \ , \tag{70}$$

$$\frac{d\psi_j}{dt} = \frac{\delta_{xj}\dot{\delta}_{yj} - \delta_{yj}\dot{\delta}_{xj}}{\delta_{xj}^2 + \delta_{yj}^2} \ , \tag{71}$$

to give $\varphi_j(t)$, which is the angle to the maximum of the radial shift, and $\psi_j(t)$, which is the angle to the maximum of the azimuthal density enhancement. These angles are analogous to the line-of-nodes angles for the tilting of the rings of a disk galaxy (Lovelace 1998). Thus, we solve $10N$ first order equations. In all cases, the total energy (54) and total canonical angular momentum (58) are accurately conserved. The different frequencies Ω_j, $\Omega_{\mathcal{E}j}$, and $\Omega_{\Delta j}$, and the tidal coefficients $\{C_{jk}\}$, etc., are evaluated using the equations of the Appendix.

6.1. *Exponential Disk*

Here, we consider the eccentric motion of the rings for the case where $M_j = 2\pi r_j \delta r \Sigma_d(r_j)$ with $\Sigma_d(r) = \Sigma_{d0}\exp(-r/r_d)$. The mass of the center is assumed given by (47), which gives $M_0 \approx 1.06 \times 10^9$ for the parameters of Figure 1. Alternatively, this value of M_0 could be due in part to a central black hole.

Figure 6 shows the dependences of the radial shifts $\epsilon_j(\varphi_j)$ and azimuthal displacements $\delta_j(\psi_j)$ of the rings ($j = 1-31$) and the radial shift of the "center" $\epsilon_0(\varphi_0)$ at a short time, 100 Myr after an initial perturbation. This type of plot is related to the plots emphasized by Briggs (1990) for characterizing the warps of galactic disks (see also Lovelace 1998). The angles φ and ψ are analogs of the line-of-nodes angle the warp.

From Figure 6 note that the azimuthal displacements δ_j are larger than the radial displacements ϵ_j so that the displacement of the center of mass of a ring is dominated by δ_j. The eccentric motion of the inner rings, say, $j = 1-4$ or $r_1 = 1$ to $r_4 = 2.5$ kpc, show the most rapid, exponential growth. For these rings the angles ψ_j and φ_j are approximately $180°$ degrees out of phase, and this agrees with the behavior found for the "second" instability of two rings discussed in §4. As mentioned, this allows the center of mass of each ring to move closer to the origin, which is a lower energy configuration. Note that with increasing j, both φ_j and ψ_j decrease for the inner rings which corresponds to a *trailing* pattern the same as found for the second instability of two rings. Note also that the shift of the "center" ϵ_0 trails the shift of the center of mass of

FIG. 6.— Polar plot of the radial shift ϵ_j and azimuthal displacement δ_j of ring matter as a function of the angles φ_j and ψ_j $(j = 1-31)$ at time $t = 100$ Myr. The radial shift of the "center" ϵ_0 is indicated by the solid dot. The conditions correspond to the galaxy parameters of Figure 1. The rings have radii $r_j = 1 + \delta r(j-1)$ kpc and $\delta r = 0.5$ kpc for $j = 1, .., 31$, masses $M_j = 2\pi r_j \delta r \Sigma_d(r_j)$ with Σ_d given in §2.1, and M_0 given by (47), which gives $M_0 \approx 1.06 \times 10^9 M_\odot$. The initial values of the shifts and displacements are $\epsilon_j = 0.1(r_j/r_{max}) = \delta_j$, $\varphi_j = 0 = \psi_j$, and $\epsilon_{0x} = 10^{-6}$ and $\epsilon_{0y} = 0$. The units of ϵ_j and δ_j are arbitrary in that the equations are linear.

the $j = 1$ ring in agreement with §5. Thus the torque of the first ring on the center acts to increase the angular momentum of the center while the torque of the center on the first ring decreases the rings angular momentum.

Figure 7 shows the exponential growth of the azimuthal displacement δ_1 and radial shift ϵ_1 of the first ring and the simultaneous growth of the radial shift of the center ϵ_0. The $e-$folding time is about 29 Myr. For comparison, the period of oscillation of the center is $T_0 = 2\pi/\Omega_0 \approx 46$ Myr for the conditions shown, where Ω_0 is given by (51). The growth of the eccentric motion of the inner rings is reduced somewhat if the mass of the center is reduced to $M_0 = 10^8 M_\odot$; the $e-$folding time for in this case is about 38 Myr for ring 1. Figure 8 shows the perturbations of the angular momentum of the center P_0 and the rings P_j at $t = 100$ Myr obtained from (58). In agreement with §5 and the abovementioned direction of the torque, the angular momentum of the center increases while that of the first and second ring decrease. The decrease in angular momenta of these rings will result in their radii shrinking. Note that the center rotates in the same direction as the disk matter.

The present linear theory does not address the issue of saturation of growth of the eccentric motion. One possibility is that the strong instability of the inner rings of the disk leads to the destruction of this part of the disk.

6.2. *Exponential Disk with Rings 1-3 Reduced*

Here, we consider the eccentric motion of the rings for the case where the ring masses are the same as in §6.1 *except* that $M_1 \to 10^{-2} M_1$, $M_2 \to 0.1 M_2$, and $M_3 \to 0.3 M_3$, which are the rings with radii $r_1 = 1$, $r_2 = 1.5$, and $r_3 = 2$ kpc. The mass of the center is the same as in §6.1, $M_0 \approx 1.06 \times 10^9 M_\odot$. The disk mass is reduced by

FIG. 7.— The plot shows the exponential growth of center shift ϵ_0 and radial shift ϵ_1 and azimuthal displacement δ_1 of the first ring at $r_1 = 1$ kpc for the same conditions as for Figure 6.

FIG. 8.— Plot of the perturbations of the angular momentum of the center P_0 and the rings P_j at $t = 100$ Myr for the same conditions as Figure 6.

a factor ≈ 0.84 compared with an exponential disk. The aim of reducing $M_1 - M_3$ is to reduce the growth rate of the eccentric motion of this part of the disk.

Figure 9 shows the essential behavior in a polar plot of the displacements and shifts at two times. The radial shift of the center is negligible on the scale of this figure. The $e-$folding time for ring 3 is about 49 Myr. Notice that the curves traced out by δ_j and by ϵ_j are approximated straight lines from the origin which rotate rigidly in the direction of motion of the matter for $j = 4$ to about $j = 11$, which corresponds to $r_3 = 2.5$ to $r_{11} = 6$ kpc. The instantaneous period of rotation of this pattern is ≈ 60 Myr, which is longer that the oscillation period at the center, $T_0 \approx 46$ Myr. This case is an example of the *phase-locking* of the eccentric motion of these rings due to the self-gravity between the rings. This phase-locking is analogous to that which occurs in the tilting motion of the rings representing a disk galaxy due to self-gravity (Lovelace 1998). In the case of tilting of rings the phase-locking results in the line-of-node angles of the rings in the inner part of the disk becoming the same.

Also in this case the growth of the eccentric instability of the inner rings is sufficiently fast that it probably further disrupts the inner part of the disk.

6.3. *Reduced Inner Disk*

Here, we study the eccentric motion of a disk the inner part of which is attenuated relative to an exponential disk. Specifically, the rings masses are $M_j = 2\pi r_j \delta r\, \hat{\Sigma}_d(r_j)$, where

$$\hat{\Sigma}_d(r) = \Sigma_{d0} \exp\left(-\beta \frac{r_d}{r}\right) \exp\left(-\frac{r}{r_d}\right) , \qquad (72)$$

with $\beta =$const. We consider $\beta = 1$. The mass of the center is the same as in §6.1, $M_0 \approx 1.06 \times 10^9 M_\odot$. However, the motion of the center is negligible and value M_0 has little influence on the eccentric motion of the disk described below. The disk mass is smaller than for an exponential disk by a factor ≈ 0.47.

Figure 10 shows the essential behavior in a polar plot of the shifts and displacements at two times. The shift of the center is negligible on the scale of the plot. The magnitude of azimuthal displacement of the outer ring δ_N exhibits an approximately *linear* growth with time, $\delta_N \approx$ const $+ t/660$Myr, for the considered initial conditions and $t \lesssim 1$ Gyr. In contrast, the magnitude of the radial shifts ϵ_j remains bounded by its initial maximum value $\epsilon_N(t = 0)$.

The patterns formed by both δ_j and ϵ_j in Figure 10 are *trailing* spirals. This is different from the proposal of Baldwin *et al.* (1980) that leading spirals should form. The rotation of the outer point on the spiral $\Delta_N(r_N = 16$ kpc) in Figure 10 is in the direction of rotation of the disk matter. Its pattern speed Ω_p corresponds to a period $2\pi/\Omega_p \approx 460$ Myr, which is a factor ≈ 1.24 longer than the rotation period of matter at this radius (≈ 370 Myr). The pattern period of say $\Delta_{15}(r = 8$kpc) is ≈ 188 Myr, which is a less than the rotation period of the matter at this radius (≈ 206 Myr). Thus, it is evident that the spiral is "wrapping up" as time increases. At the same time, the spiral pattern propagates radially outward. The outward speed is about 10 km/s at $r \sim 8$ kpc for $t \sim 300$ Myr.

Figure 11 shows a polar plot of the radius to the maximum of the azimuthal displacement $r_j(\psi_j)$ at two times. The curve is a trailing spiral with an approximate fit given by

$$\psi = A(t) \exp\left(-\frac{r}{a(t)}\right) , \qquad (73)$$

where $A \approx 0.065\ (t/\mathrm{kpc})$ rad, and $a \approx 7.0{+}0.0044\ (t/\mathrm{Myr})$ kpc for $t \lesssim 1$ Gyr. Thus the radial spacing between spiral arms is $\lambda_r \approx (2\pi a/A) \exp(r/a)$ for $\lambda_r \ll a$. For validity of the ring representation we must have $\lambda_r \geq 2\delta r (= 1$ kpc), where δr the separation between rings.

Figure 12 shows the radial variation of the perturbations in ring angular momenta. The perturbation of the angular momentum of the central mass is negligible. This figure should be compared with Figure 8 which gives the same plot for an exponential disk.

Some simplification of equations is possible in the present case at long times due to the fact that $\delta_j \gg \epsilon_j$. Firstly, we have $\delta \mathbf{v} \approx \delta v_\phi \hat{\boldsymbol{\phi}}$ with

$$\delta v_\phi(r, \phi) \approx -(\dot{\delta}_x + \Omega \delta_y) \sin\phi + (\dot{\delta}_y - \Omega \delta_x) \cos\phi \ . \quad (74)$$

Secondly,

$$\frac{\delta\Sigma(r,\phi)}{\Sigma(r)} \approx \frac{1}{r} (\delta_x \cos\phi + \delta_y \sin\phi) \ , \qquad (75)$$

for $\delta_j \gg r\epsilon_j/r_d$.

Figure 13 shows the profiles along the $x-$axis through the galaxy center of the fractional change in the surface density $\delta\Sigma/\Sigma$ and the change in the azimuthal velocity δv_ϕ obtained from (74) and (75). The opposite signs of δv_ϕ on the two sides of the galaxy would of course make the rotation curves on the two sides different as observed in some cases (Swaters *et al.* 1998). Note that in some regions the changes $\delta\Sigma$ and δv_ϕ are correlated and in other regions they are anticorrelated. Figure 14 shows two-dimensional appearance of the fractional surface density variations from (75).

For long times $t > 1$ Gyr, the the azimuthal displacements and shifts of the inner rings (2 and 3) start to become large compared with the values in the outer disk ($r > 4$ kpc) even though these rings have very small masses. At the same time, the displacement of the center, which has mass $M_0 = 1.06 \times 10^9 M_\odot$, grows, and at $t = 1$ Gyr it is $\epsilon_0 \approx 0.057$ for the conditions of Figures 9-13. If the mass of the center is $M_0 = 10^6 M_\odot$, then the displacements and shifts of rings 2 and 3 at $t = 1$ Gyr are significantly reduced as is the shift of the center which is $\epsilon_0 \approx 0.014$.

7. CONCLUSIONS

The paper develops a theory of eccentric ($m = \pm 1$) linearized perturbations of an axisymmetric disk galaxy residing in a spherical dark matter halo and with a spherical bulge component. The disk is represented by a large but finite number N of rings with shifted centers *and* with perturbed azimuthal matter distributions. This description is appropriate for a disk with small 'thermal' velocity spread v_{th} where the matter is in approximately laminar circular motion. The spread for a thin disk has $(v_{th}/v_\phi)^2 \ll 1$, but it is sufficent to give a Toomre $Q(r) \gtrsim 1$. Earlier, Baldwin *et al.* (1980) discussed asymmetries in disk galaxies in terms of shifted rings but without interactions between the rings and without azimuthal displacements of the ring matter. Account is taken of the shift of the matter at the galaxy's center, which may include a massive black hole. The gravitational interactions between the rings and between the rings and the center is fully accounted for, but the halo and bulge components are treated as passive gravitational field sources. Equations of motion are derived for the ring and the center, and from these we obtain the Lagrangian for the rings+center system. For this system we derive an energy constant of the motion, and a total canonical angular momentum constant of the motion.

We first discuss the nature of the precession of a single ring with the other rings fixed; this case although not self-consistent is informative. There are four modes, analogs to the normal modes of a non-rotating system, and two have negative energy and two positive energy. Negative energy modes are unstable in the presence of dissipation such as that due to dynamical friction. We go on to study the eccentric motion of a disk consisting of two rings of different radii but equal mass $M_d/2$. Above a threshold value of M_d the two rings are unstable with instability due merging of positive and negative energy modes. This result is obtained by solving the eighth order polynomial for the frequencies of the eight modes. Above a second, somewhat larger threshold value of M_d, a second instability appears, and in this case the ring motion is such that the angular momentum of the inner ring decreases while that of the outer ring increases. For the unstable motion, the maximum of the azimuthal density enhancement of a ring occurs at an angle about 180° from the direction of the radial shift. This allows the center of mass of the ring to move closer to the center of mass of the other ring and

FIG. 9.— Polar plot of the radial shift ϵ_j and azimuthal displacement δ_j of ring matter as a function of the angles φ_j and ψ_j of the maxima of the shift and displacement at times $t = 300$ and 305 Myr. The solid dot labeled by the arrow indicates the shift of the center at $t = 300$ Myr; the other dot is the shift at 305 Myr. The rings, the center, and the initial values of the shifts and displacements are the same as in Figure 6 except that $M_1 \to 10^{-2} M_1$, $M_2 \to 0.03 M_2$, and $M_3 \to 0.1 M_3$. Thus the conditions correspond to the galaxy parameters of Figure 1 except that the disk mass is reduced to $\approx 5.06 \times 10^{10} M_\odot$. The shifts and displacements of the rings $j = 1, 2$ are dynamically unimportant and are not shown.

Fig. 10.— Polar plot of the radial shift ϵ_j and azimuthal displacement δ_j of ring matter as a function of the angles φ_j and ψ_j of the maxima of the shift and displacement at times $t = 300$ and 400 Myr. The radial shift of the center is negligible on the scale of this plot. The rings, the center, and the initial values of the shifts and displacements are the same as in Figure 6 except that the ring masses are obtained from (72). The mass of the center is the same as in Figure 6, $M_0 \approx 1.06 \times 10^9 M_\odot$. Thus the conditions correspond to the galaxy parameters of Figure 1 except that the disk mass is reduced to $\approx 2.82 \times 10^{10} M_\odot$. The shifts and displacements of the rings $j = 1, 2$ are dynamically unimportant and are not shown.

to the origin.

We also analyze the eccentric motion of a disk of one ring interacting with a radially shifted central mass. This system has six modes, the frequencies of which are obtained by solving a sixth order polynomial. In this case, instability sets in above a threshold value of the central mass (for a fixed ring mass), and it acts to increase the angular momentum of the central mass (which therefore rotates in the direction of the disk matter), while decreasing the angular momentum of the ring. The instability is again due to the merging of positive and negative energy modes.

We study the eccentric dynamics of a disk with an exponential surface density distribution represented by a large number $N = 31$ of rings and a central mass $M_0 \sim 10^9 M_\odot$ which may include the mass of a black hole. The outer radius of the disk is $r_N = 16$ kpc; we have checked that this value has negligible affect on the reported results. In this case, we numerically integrate the equations of motion. A check on the validity of the integrations is provided by

Fig. 11.— Polar plot of the radius to the maximum of the azimuthal displacement $r_j(\psi_j)$ at two times for the same case as Figure 10.

Fig. 12.— Radial dependence of the perturbations in the ring angular momenta P_j at $t = 300$ Myr from equation (58). The angular momentum of the center P_0 is negligible.

monitoring the mentioned total energy and total canonical angular momentum, which are found to be accurately constant in all presented results. The inner part of the disk $r \lesssim 2.5$ kpc is found to be strongly unstable with $e-$folding time ~ 30 Myr for the conditions considered. The $e-$folding time is somewhat longer if $M_0 = 0$. Angular momentum of the rings is transferred *outward*, *and* to the central mass if it is present. A *trailing* one-armed spiral wave is formed in the disk. This differs from the prediction of Baldwin *et al.* (1980) of a leading one-armed spiral. The outer part of the disk $r \gtrsim r_d$ is stable and in this region the angular momentum is transported by the wave. Thus our results appear compatible with the theorem of Goldreich and Nicholson (1989) regarding angular momentum in *stable* rotating fluids. The instability found here appears qualitatively similar to that found by Taga and Iye (1998b) for a fluid Kuzmin disk with surface density $\Sigma \propto 1/(1+r^2)^{3/2}$ with a point mass at the center where unstable trailing one-armed spiral waves are found.

The present linear theory does not address the issue of saturation of growth of the eccentric motion. One possibility is that the strong instability of the inner rings of the disk leads to the destruction of this part of the disk. For this reason we have studied a disk with a modified exponential density distribution where the surface density of the inner part of the disk is reduced. However, the mass of the center of the galaxy was kept the same as in the case of an exponential disk, $M_0 \sim 10^9 M_\odot$. In this case we find much slower, linear - as opposed to exponential - growth of the eccentric motion of the disk for times $t \lesssim 1$ Gyr. A trailing one-armed spiral wave forms in the disk and becomes more tightly wrapped as time increases. Angular momentum is transferred outward. The motion of the central mass if present is small compared with that of the disk for $t \lesssim 1$ Gyr.

For long times $t > 1$ Gyr, the the azimuthal displacements and shifts of the inner rings start to become large compared with the values in the outer disk. At the same time, the radial shift of the center grows. This shift is significantly reduced if the mass of the center is changed from $\sim 10^9$ to $10^6 M_\odot$.

We thank M.S. Roberts for valuable discussions which stimulated this work. We thank R.H. Miller for bringing the work of Taga and Iye to our attention and for valuable comments on our work. We thank M.M. Romanova for

Fig. 13.— Profiles of fractional density variations $10^2 \delta\Sigma(x,0)/\Sigma(x,0)$ (in percent) and variation of azimuthal velocity $10^3 \delta v_y(x,0)/v_\phi(N)$ along the $x-$axis through the middle of the galaxy at $t = 400$ Myr for the same case as Figure 10. Here, $v_\phi(N) \approx 265$ km/s is the disk rotation velocity at $r = 16$ kpc. The vertical scale is arbitrary in that the equations solved are linear, but the ratio $\delta\Sigma/\delta v_\phi$ is fixed.

FIG. 14.— Two-dimensional appearance of the fractional density variations $10^2\delta\Sigma(x,y)/\Sigma(x,y)$ at $t = 400$ Myr for the same case as Figure 13.

valuable discussions and an anonymous referee for valuable comments. This research has been partially supported by NSF grant AST 95-28860 to M.P.H. and NASA grant NAG5 6311 to R.V.E.L.

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APPENDIX

TIDAL COEFFICIENTS

For $|r_j - r_k| \gg \sqrt{\Delta r_j \Delta r_k}$, the ring profiles can be treated as delta functions, $S(r|r_j) \to \delta(r-r_j)/r$, and consequently the 'tidal coefficients' of equations (29) - (32) can be simplified to give

$$\begin{aligned}
C_{jk} &\approx GM_jM_k \frac{\partial^2 \mathcal{K}(r_j,r_k)}{\partial r_j \, \partial r_k} = -\frac{2GM_jM_k}{\pi(r_j+r_k)(r_j-r_k)^2} \, E(k_{jk}) \ , && \text{(A1)}\\
D_{jk} &\approx GM_jM_k \frac{\partial \mathcal{K}(r_j,r_k)}{r_k \, \partial r_j} = -\frac{GM_jM_k}{\pi r_j^2(r_j^2-r_k^2)}\Big[(r_j+r_k)E(k_{jk}) + (r_j-r_k)K(k_{jk})\Big] \ , && \text{(A2)}\\
D'_{jk} &\approx GM_jM_k \frac{\partial \mathcal{K}(r_j,r_k)}{r_j \, \partial r_k} = D_{kj} \ , && \text{(A3)}\\
E_{jk} &\approx GM_jM_k \frac{\mathcal{K}(r_j,r_k)}{r_j \ r_k} = \frac{GM_jM_k}{\pi r_j^2 r_k^2 (r_j+r_k)}\Big[(r_j^2+r_k^2)K(k_{jk}) - (r_j+r_k)^2 E(k_{jk})\Big] \ , && \text{(A4)}
\end{aligned}$$

(Mathematica V. 3) where $k_{jk} \equiv 2\sqrt{r_j r_k}/(r_j+r_k)$, where $\mathcal{K}$ is defined by equation (31), and where

$$K(k) \equiv \int_0^{\pi/2} d\phi \Big/ \sqrt{1-k^2\sin^2\phi} \ , \quad \text{and} \quad E(k) \equiv \int_0^{\pi/2} d\phi \sqrt{1-k^2\sin^2\phi} \ ,$$

are complete elliptic integrals of the first and second kinds respectively. Note that for $r_k \gg r_j$, $C_{jk} \approx -GM_jM_k/r_k^3$, $D_{jk} \approx GM_jM_k/(2r_k^3)$, $D'_{jk} \approx -GM_jM_k/r_k^3$, and $E_{jk} \approx GM_jM_k/(2r_k^3)$.

In the opposite limit where $|r_j - r_k| \ll \sqrt{r_j r_k}$, equations (27)-(30) can be evaluated approximately as

$$\begin{aligned}
C_{jk} &\approx \frac{GM_jM_k}{8\pi\sqrt{\pi}(\Delta r)\bar{r}^2} \int_{-\infty}^{\infty} \frac{ydy \ \exp(-y^2/4)}{r_k - r_j + y\Delta r}\left\{2\bar{r} - (r_k - r_j + y\Delta r)\ln\left[\frac{8\bar{r}}{|r_k - r_j + y\Delta r|}\right]\right\} \ ,\\
&\approx \frac{GM_jM_k}{2\pi(\Delta r)^2\bar{r}}\left[1 - \frac{\sqrt{\pi}}{2} u \exp(-u^2/4)\mathrm{erfi}(u/2) + \mathcal{O}(\Delta r/\bar{r})\right] \ , && \text{(A5)}
\end{aligned}$$

where $\bar{r} \equiv (r_j + r_k)/2$, $\Delta r \equiv \sqrt{(\Delta r_j^2 + \Delta r_k^2)/2}$, $u \equiv (r_k - r_j)/\Delta r$, and and $\mathrm{erfi}(x) \equiv \mathrm{erf}(ix)/i = (2/\sqrt{\pi})\int_0^x dy \exp(y^2)$. The integral in (A5) is a principal value integral of the form occurring in the plasma dispersion function W (Ichimaru 1973). Also,

$$\begin{aligned}
D_{jk} &\approx \frac{GM_jM_k}{4\pi\sqrt{\pi} \ \bar{r}^3} \int_{-\infty}^{\infty} \frac{dy \ \exp(-y^2/4)}{r_k - r_j + y\Delta r}\left\{2\bar{r} - (r_k - r_j + y\Delta r)\ln\left[\frac{8(\bar{r}/\Delta r)}{|u+y|}\right]\right\} \ ,\\
&\approx -\frac{GM_jM_k}{4\pi\sqrt{\pi} \ \bar{r}^3}\left\{\int_{-\infty}^{\infty} dy \ \exp(-y^2/4)\ln\left[\frac{8\bar{r}/\Delta r}{|u+y|}\right] - 2\pi(\bar{r}/\Delta r)\exp(-u^2/4)\mathrm{erfi}(u/2)\right\} \ , && \text{(A6)}
\end{aligned}$$

and

$$E_{jk} \approx \frac{GM_jM_k}{2\pi\sqrt{\pi}\bar{r}^3} \int_{-\infty}^{\infty} dy \exp(-y^2/4)\ln\left[\frac{1.0827(\bar{r}/\Delta r)}{|u+y|}\right] \ . \qquad \text{(A7)}$$

From these expressions we obtain

$$C_{jj} \approx \frac{GM_j^2}{2\pi r_j(\Delta r_j)^2} \ , \quad D_{jj} \approx -\frac{GM_j^2}{2\pi r_j^3}\ln\left(\frac{10.68 r_j}{\Delta r_j}\right) \ , \quad E_{jj} \approx \frac{GM_j^2}{\pi r_j^3}\ln\left(\frac{1.445 r_j}{\Delta r_j}\right) \ . \qquad \text{(A8)}$$

Equations (A1) - (A8) are valuable for numerical evaluation of the tidal coefficients.

In the following we derive some useful relations involving the tidal coefficients. From equation (12) we have

$$\Omega_d^2(r) = \frac{1}{r}\frac{\partial \Phi_d}{\partial r} = 2\pi G \int_0^{\infty} r' dr' \Sigma_d(r') \oint \frac{d\Psi}{2\pi} \frac{(1 - r'\cos\Psi/r)}{[r^2 + (r')^2 - 2rr'\cos\Psi]^{3/2}} \ . \qquad \text{(A9)}$$

The Ψ integral in this case is

$$\frac{1}{\pi r^2[r^2 - (r')^2]}\left[(r + r')E + (r - r')K\right] \ . \qquad \text{(A10)}$$

Comparison with (A2) shows that shows that

$$\Omega_{dj}^2 \equiv \int_0^{\infty} rdrS(r|r_j)\Omega_d^2(r) = -\frac{1}{M_j}\sum_{k=1}^{N} D_{jk} \ . \qquad \text{(A11)}$$

This expression does not include the disk mass within the inner ring. As discussed in §2.6, this part of the disk is treated as a point mass M_0 with unperturbed position $\mathbf{r} = 0$. If there is a central black hole M_{bh}, its mass is include in M_0. To account for the influence of M_0, we simply add the term GM_0/r_j^3 to the right hand side of (A11). The resulting expression for Ω_{dj} is useful for the numerical calculations. For $N \gtrsim 30$ and $r_j = 1$ to $10 - 20$ kpc, we find that equation (A11) gives accurate agreement with the analytic expression (3).

An alternative expression for Ω_d^2 can be obtained by integration by parts,

$$\Omega_d^2(r) = -\,\frac{2\pi G}{r}\int_0^\infty r'dr'\frac{\partial \Sigma'}{\partial r'}\;\mathcal{K}(r,r')\;. \tag{A12}$$

Thus

$$M_j\left(r\frac{\partial \Omega_d^2}{\partial r}\right)_j = \int rdrS(r|r_j)r\frac{\partial \Omega_d^2(r)}{\partial r} = GM_j\sum_k M_k \iint rdr\;r'dr'S(r|r_j)\frac{\partial S(r'|r_k)}{\partial r'}\frac{\partial[\mathcal{K}(r,r')/r]}{\partial r}\;,$$

or

$$M_j\left(r\frac{\partial \Omega_d^2}{\partial r}\right)_j = GM_j\sum_k M_k\int\int rdr\;r'dr'S(r|r_j)\left[\frac{1}{r'}\frac{\partial[r'S(r'|r_k)]}{\partial r'} - \frac{S(r'|r_k)}{r'}\right]\left[\frac{\mathcal{K}(r,r')}{r} - \frac{\partial \mathcal{K}(r,r')}{\partial r}\right] \tag{A13}$$

In view of equations (27) - (30), this equation can be written as

$$M_j\left(r\frac{\partial \Omega_d^2}{\partial r}\right)_j = \sum_k (C_{jk} + D_{jk}) - \sum_k (D'_{jk} + E_{jk})\;. \tag{A14}$$

This relation is useful in §2.5.

We also evaluate the disk gravitational potential in the ring representation,

$$\begin{aligned}\Phi_d(r) &= -G\int d^2r'\;\frac{\Sigma_d(r')}{|\mathbf{r}-\mathbf{r}'|} = -\;2\pi G\int_0^\infty r'dr'\;\Sigma_d(r')\oint\frac{d\Psi}{2\pi}\frac{1}{[r^2+(r')^2-2rr'\cos\Psi]^{1/2}} \\ &= -2\pi G\int r'dr'\;\Sigma_d(r')\frac{2K(k)}{\pi(r+r')}\;.\end{aligned} \tag{A15}$$

With

$$\Phi_{dj} \equiv \int_0^\infty rdr\;S(r|r_j)\Phi_d(r) = \frac{1}{M_j}\sum_k \Lambda_{jk}\;, \tag{A16}$$

where

$$\Lambda_{jk} = -GM_jM_k\int\int rdr\;r'dr'\;S(r|r_j)S(r'|r_k)\;\frac{2K(k)}{\pi(r+r')}\;. \tag{A17}$$

For $|r_j - r_k| \gg \sqrt{\Delta r_j \Delta r_k}$,

$$\Lambda_{jk} \approx -GM_jM_k\frac{2K(k)}{\pi(r_j+r_k)}\;, \tag{A18}$$

whereas for $|r_j - r_k| \ll \sqrt{r_j r_k}$

$$\Lambda_{jk} \approx -GM_jM_k\frac{1}{2\pi\sqrt{\pi}\;\bar{r}}\int_{-\infty}^\infty dy\exp(-y^2/4)\ln\left[\frac{8\bar{r}/\Delta r}{|u+y|}\right]\;, \tag{A19}$$

where $u \equiv (r_k - r_j)/\Delta r$ as above. Note that $\Lambda_{jk} = \Lambda_{kj}$ and that $\Lambda_{jj} \approx -GM_j^2(1/\pi r_j)\ln(10.68 r_j/\Delta r_j)$.